

Harnack inequalities and Bounds for Densities of Stochastic Processes

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Dedicated to Valentin Konakov in occasion of his 70th birthday.

Abstract

We consider possibly degenerate parabolic operators in the form

$$\mathcal{L} = \sum_{k=1}^m X_k^2 + X_0 - \partial_t,$$

that are naturally associated to a suitable family of stochastic differential equations, and satisfying the Hörmander condition. Note that, under this assumption, the operators in the form \mathcal{L} has a smooth fundamental solution that agrees with the density of the corresponding stochastic process. We describe a method based on Harnack inequalities and on the construction of Harnack chains to prove lower bounds for the fundamental solution. We also briefly discuss PDE and SDE methods to prove analogous upper bounds. We eventually give a list of meaningful examples of operators to which the method applies.

1 Introduction

Let $(W_t)_{t \geq 0}$ denote a m -dimensional Brownian motion, $W_t = (W_t^1, \dots, W_t^m)$. It is well known that the N -dimensional solution of the SDE

$$dZ_t^i = \sum_{j=1}^m \sigma_{ij}(t, Z_t) \circ dW_t^j + b_i(t, Z_t) dt, \quad i = 1, \dots, N, \quad t \geq 0 \quad (1.1)$$

is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by $(W_t)_{t \geq 0}$. Here “ $\circ dW_t$ ” stands for the Stratonovich integral. We denote by $Z_t^{x_0}$ the

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solution of the SDE (1.1) with initial condition $Z_0^{x_0} = x_0$. The equation (1.1) is associated to the Kolmogorov operator

$$\mathcal{L} = \sum_{i=1}^m X_i^2 + X_0 - \partial_t$$

where

$$X_i(t, x) = \frac{1}{\sqrt{2}} \sum_{j=1}^m \sigma_{ij}(t, x) \partial_{x_j}, \quad i = 1, \dots, m, \quad X_0(t, x) = \sum_{i=1}^N b_i(t, x) \partial_{x_i}. \quad (1.2)$$

We introduce the $N \times N$ matrix $A(t, x) = (a_{ij}(t, x))_{i,j=1,\dots,N}$ whose elements are

$$a_{ij}(t, x) = \frac{1}{2} \sum_{k=1}^m \sigma_{ik}(t, x) \sigma_{jk}(t, x), \quad i, j = 1, \dots, N,$$

and we note that

$$\langle A(x, t) \xi, \xi \rangle = \frac{1}{2} \|\sigma(t, x) \xi\|^2 \geq 0. \quad \text{for every } \xi \in \mathbb{R}^N.$$

If the smallest eigenvalue of $A(t, x)$ is uniformly positive we say that the operator \mathcal{L} is uniformly parabolic. A keystone result in the theory of parabolic partial differential equations reads as follows. Assume that there exist two positive constants λ, Λ such that

$$\lambda |\xi|^2 \leq \langle A(x, t) \xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \text{for every } x \in \mathbb{R}^N, \text{ and } \xi \in \mathbb{R}^N, \text{ for all } t \in]0, T]. \quad (1.3)$$

If $\Gamma = \Gamma(x, t, \xi, \tau)$ denotes the fundamental solution of the PDE

$$\partial_t u(x, t) = \sum_{i,j=1}^N \partial_{x_i} (a_{ij}(x, t) \partial_{x_j} u(x, t)), \quad (x, t) \in \mathbb{R}^N \times]0, T], \quad (1.4)$$

then there exist positive constants c^-, C^-, c^+, C^+ such that

$$\frac{c^-}{(t - \tau)^{N/2}} \exp \left(-C^- \frac{|x - \xi|^2}{t - \tau} \right) \leq \Gamma(x, t, \xi, \tau) \leq \frac{C^+}{(t - \tau)^{N/2}} \exp \left(-c^+ \frac{|x - \xi|^2}{t - \tau} \right), \quad (1.5)$$

for every $(x, t), (\xi, \tau) \in \mathbb{R}^N \times]0, T]$ with $\tau < t$. This upper bound has been proved by Aronson [1] for operators with bounded measurable coefficients a_{ij} 's, while the lower bound has been proved by Moser [27, 28]. We also refer to the fundamental work of Nash [29] for a related result, and to Krylov and Safonov [21] for non-degenerate form operators.

The results described above have been extended by several authors to possibly degenerate operators in the form

$$\mathcal{L} := \sum_{k=1}^m X_k^2 + Y, \quad Y := X_0 - \partial_t, \quad (1.6)$$

where X_0, X_1, \dots, X_m are smooth vector fields on \mathbb{R}^{N+1} , that is

$$X_i(x, t) = \sum_{j=1}^N c_{i,j}(x, t) \partial_{x_j}, \quad i = 0, \dots, m. \quad (1.7)$$

for some smooth functions $c_{i,j}$'s. In particular, upper bounds have been proved by a PDE approach that goes back to Aronson's work [1], or by an approach based on Lyapunov functions (see [25] and the references therein). Several authors prove bounds analogous to (1.5) in the framework of stochastic processes. We refer to the works of Malliavin [23], Kusuoka and Stroock [22], where a general method to prove upper bounds for density is introduced, and to the work of Ben Arous and Léandre [5], where the Malliavin Calculus is further developed. We also refer to the monograph of Nualart [30] for a comprehensive presentation of this subject.

In general, lower bounds have been proved by following the idea introduced by Moser in [27]. In this note we briefly describe this method for uniformly parabolic partial differential equations, then we give an overview of more recent articles where it has been adapted to the study of degenerate parabolic equations in the form (1.6). This idea is also used in the works where lower bounds are proved by probabilistic methods. We refer to Kohatsu-Higa [20], Bally [3], Bally and Kohatsu-Higa [4].

We now give a list of examples of operators considered in this note. Each one of them is the prototype of a wide family of differential operators.

- *Heat operator on the Heisenberg group* $\mathcal{L} = X_1^2 + X_2^2 - \partial_t$, where

$$X_1 = \partial_x - \frac{1}{2}y\partial_w, \quad X_2 = \partial_y + \frac{1}{2}x\partial_w.$$

Note that \mathcal{L} acts on the variable $(x, y, w, t) \in \mathbb{R}^4$, and writes in the form (1.6) with $X_0 = 0$. The degenerate elliptic operator $\Delta_{\mathbb{H}} = X_1^2 + X_2^2$ is said *sub-Laplacian* on the Heisenberg group.

- *Kolmogorov Operator* $\mathcal{L} = \partial_{xx} + x\partial_y - \partial_t, (x, y, t) \in \mathbb{R}^3$. In this case $\mathcal{L} = X^2 + Y$ with $X = \partial_x, Y = x\partial_y - \partial_t$.
- *More Degenerate Kolmogorov Operators* $\mathcal{L} = \partial_{xx} + x^2\partial_y - \partial_t, (x, y, t) \in \mathbb{R}^3$. In this case $\mathcal{L} = X^2 + Y$ with $X = \partial_x, Y = x^2\partial_y - \partial_t$.
- *Asian Option Operator* $\mathcal{L} = x^2\partial_{xx} + x\partial_x + x\partial_y - \partial_t, (x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2$. In this case $\mathcal{L} = X^2 + Y$ with $X = x\partial_x, Y = x\partial_y - \partial_t$.

All the operators in the above list are strongly degenerate, since the smallest eigenvalue of the characteristic form is zero for all the above examples. In general, operators in the form (1.6) cannot be uniformly parabolic if $m < N$. On the other hand, all the examples do satisfy the following condition:

HYPOTHESIS [H] $\mathcal{L} = \sum_{k=1}^m X_k^2 + Y$ satisfies the Hörmander condition if

$$\text{rank} (\text{Lie}\{X_1, \dots, X_m, Y\}(x, t)) = N + 1, \quad \text{for every } (x, t) \in \mathbb{R}^{N+1}.$$

In the sequel we only consider operators \mathcal{L} satisfying the Hörmander condition. It is known that, for this family of operators, the law of the stochastic process (1.1) is absolutely continuous with respect to the Lebesgue measure in \mathbb{R}^N , and that its density is smooth. Moreover, the density $p(\xi, \tau; x, t)$ is linked with the fundamental solution $\Gamma(x, t; x_0, t_0)$ of \mathcal{L} by the relation

$$p(\xi, \tau; x, t) = \Gamma(x, T - t; \xi, T - \tau).$$

It is also known that the regularity properties of the operators satisfying the Hörmander condition are related to a Lie group structure that replaces the usual Euclidean one. In the proof of the lower bounds for positive solutions the geometric aspects of this *non Euclidean* structure will be explicitly used. To make clear the exposition, in Section 2 we recall the method used by Moser in [27] to prove the lower bound in (1.5) for uniformly parabolic operators. In Section 3 we describe how the method outlined in Section 2 is adapted to the degenerate ones, satisfying the Hörmander condition [H]. The remaining Sections 4, 5, 6 and 7 are devoted to the examples listed above.

2 Uniformly parabolic equations

In this Section we describe the method introduced by Moser [27] to prove the lower bound (1.5) of the fundamental solution for uniformly parabolic equations. The main ingredient of the method is the *parabolic Harnack inequality*, first proved by Hadamard [15] and, independently, by Pini [31] in 1954 for the heat equation, then by Moser [27, 28] for uniformly parabolic equations in divergence form (1.4). Its statement requires some notation (see Fig. 1). Let

$$Q_r(x, t) = B(x, r) \times]t - r^2, t[,$$

denote the parabolic cylinder whose upper basis is centered at (x, t) . Let $\alpha, \beta, \gamma, \delta \in]0, 1[$ be given constants, with $\alpha < \beta < \gamma < 1$,

$$Q_r^-(x, t) = B(x, \delta r) \times]t - \gamma r^2, t - \beta r^2[\quad Q_r^+(x, t) = B(x, \delta r) \times]t - \alpha r^2, t[.$$

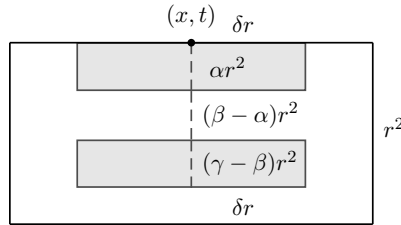


FIG. 1 - PARABOLIC HARNACK INEQUALITY.

Theorem 2.1 (PARABOLIC HARNACK INEQUALITY) *Let $Q_r(x, t) \subset \mathbb{R}^{N+1}$, and let $\alpha, \beta, \gamma, \delta \in]0, 1[$ be given constants, with $\alpha < \beta < \gamma < 1$. Then there exists $C = C(\alpha, \beta, \gamma, \delta, \lambda, \Lambda, N)$ such that*

$$\sup_{Q_r^-(x, t)} u \leq C \inf_{Q_r^+(x, t)} u$$

for every $u : Q_r(x, t) \rightarrow \mathbb{R}, u \geq 0$, satisfying (1.4). Here λ, Λ are the constants in (1.3).

Remark 2.2 *Note that C does not depend on the point (x, t) and on r , then the Harnack inequality is invariant with respect to the Euclidean translation $(x, t) \mapsto (x + x_0, t + t_0)$, and to the parabolic dilation $(x, t) \mapsto (rx, r^2t)$. For this reason, the above statement is often referred to as invariant Harnack inequality.*

In the sequel we will use the following version of the parabolic Harnack inequality (see Fig. 2). For any given $c \in]0, 1[$ we denote by

$$P_r(x, t) = \{(y, s) \in Q_r(x, t) \mid 0 < t - s \leq cr^2 < t, |y - x|^2 \leq t - s\}.$$

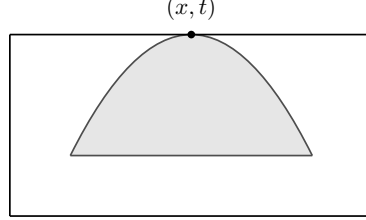


FIG. 2 - PARABOLIC HARNACK INEQUALITY.

Corollary 2.3 *Let $Q_r(x, t) \subset \mathbb{R}^{N+1}$, and let $c \in]0, 1[$ be a given constant. Then there exists $C = C(c, \lambda, \Lambda, N)$ such that*

$$\sup_{P_r(x, t)} u \leq Cu(x, t)$$

for every $u : Q_r(x, t) \rightarrow \mathbb{R}, u \geq 0$, satisfying (1.4). Here λ, Λ are the constants in (1.3).

PROOF. For every positive ρ we denote

$$S_\rho(x, t) = B(x, \rho) \times \{t - \rho^2\}.$$

Let $\alpha, \beta, \gamma \in]0, 1[$ be such that $\alpha < \beta \leq c \leq \gamma < 1$, and let $\delta = \sqrt{c}$. Then, for every $\rho \in [0, r]$ we have that u is a non-negative solution of (1.4) in the domain $Q_\rho(x, t)$. Since $S_\rho(x, t) \subset Q_\rho^-(x, t)$, from Theorem 2.1 we obtain

$$\sup_{S_\rho(x, t)} u \leq \sup_{Q_\rho^-(x, t)} u \leq C \inf_{Q_\rho^+(x, t)} u \leq Cu(x, t),$$

and the conclusion follows from the fact that $P_r(x, t) = \cup_{0 < \rho \leq r} S_\rho(x, t)$. □

With Corollary 2.3 in hand, we can easily obtain the following *non local* Harnack inequality, first proved by Moser (Theorem 2 in [27]). We also refer to Aronson & Serrin [2] for more general uniformly parabolic differential operators.

Theorem 2.4 *Let $u : \mathbb{R}^N \times]0, T[\rightarrow \mathbb{R}$ be a non-negative solution of the parabolic equation (1.4). Then there exists a positive constant $C = C(c, \lambda, \Lambda, N)$ such that*

$$u(x, t) \leq C^{1 + \frac{|x_0 - x|^2}{t_0 - t}} u(x_0, t_0),$$

for every $(x_0, t_0), (x, t) \in \mathbb{R}^N \times]0, T[$ with $t_0 - t < ct_0$.

PROOF. Let $(x_0, t_0), (x, t) \in \mathbb{R}^N \times]0, T[$, with $t_0 - t < ct_0$, choose $r = \sqrt{t}$ and note that the cylinder $Q_r(x_0, t_0)$ is contained in $\mathbb{R}^N \times]0, T[$. If $(x, t) \in P_r(x_0, t_0)$ we simply apply Corollary 2.3 and the proof is complete. If otherwise $(x, t) \notin P_r(x_0, t_0)$, we consider the segment whose end points are (x_0, t_0) and (x, t) , and denote by (x_1, t_1) the point where it intersects the boundary of $P_r(x_0, t_0)$. Note that $t_1 \geq t > (1 - c)t_0$, then (x_1, t_1) belongs to the lateral part of the boundary of $P_r(x_0, t_0)$. By Corollary 2.3 we have

$$u(x_1, t_1) \leq Cu(x_0, t_0).$$

We then iterate the argument. We define a finite sequence (x_j, t_j) , with $j = 2, \dots, k$ such that (x_j, t_j) belonging to the boundary of $P_r(x_{j-1}, t_{j-1})$ for $j = 2, \dots, k$, and $(x, t) \in P_r(x_k, t_k)$ (see Fig. 3). By applying k times Corollary 2.3 we then find

$$u(x, t) \leq Cu(x_k, t_k) \leq C^2u(x_{k-1}, t_{k-1}) \leq \dots \leq C^{k+1}u(x_0, t_0).$$

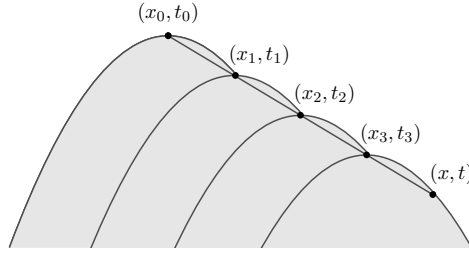


FIG. 3 - HARNACK CHAIN.

To conclude the proof it is sufficient to note that the integer k only depends on the slope of the line connecting (x_0, t_0) to (x, t) and that a simple computation gives $k < \frac{|x_0 - x|^2}{t_0 - t}$. \square

The set $\{(x_0, t_0), (x_1, t_1), \dots, (x_k, t_k), (x, t)\}$ appearing in the above proof is often referred to as *Harnack chain*. By using the following property of the fundamental solution Γ of the differential operator appearing in (1.4)

$$\Gamma(0, t) \geq \frac{C_T}{t^{N/2}}, \quad \text{for every } t \in]0, T],$$

for some positive constant $C_T = C_T(\lambda, \Lambda, N, T)$ and by choosing $c = \frac{1}{2}$ in Theorem 2.4, we conclude that there exist two positive constants C^-, c^- such that

$$\Gamma(x, t, y, s) \geq \frac{C^-}{(t - s)^{N/2}} \exp\left(-c^- \frac{|x - y|^2}{t - s}\right),$$

for every $(x, t), (y, s) \in \mathbb{R}^{N+1}$ with $0 < t - s \leq T$.

Remark 2.5 Before considering degenerate parabolic operators, we point out that the method used in the proof of Theorem 2.4 only relies on the following two ingredients.

- i) The invariance with respect to the Euclidean translation and to the parabolic dilation $(x, t) \mapsto (x_0 + \rho x, t_0 + \rho^2 t)$ are the properties that allows us to obtain Corollary 2.3 from Theorem 2.1.
- ii) Segments are very simple supports for the construction of Harnack chains. In the study of degenerate parabolic operators a more sophisticated construction will be needed.

3 Degenerate hypoelliptic operators

Consider a linear second order differential operator in the form (1.6)

$$\mathcal{L} = \sum_{k=1}^m X_k^2 + X_0 - \partial_t.$$

satisfying the Hörmander condition [H]. We introduce a definition based on the vector fields X_1, \dots, X_m, Y .

Definition 3.1 *We say that γ is an \mathcal{L} -admissible path starting from $z_0 \in \mathbb{R}^{N+1}$ if it is an absolutely continuous solution of the following ODE*

$$\begin{aligned} \dot{\gamma}(\tau) &= \sum_{k=1}^m \omega_k(\tau) X_k(\gamma(\tau)) + Y(\gamma(\tau)) \\ \gamma(0) &= z_0. \end{aligned}$$

with $\omega_1, \dots, \omega_m \in L^1([0, T])$.

Let Ω be an open subset of \mathbb{R}^{N+1} and $z_0 \in \Omega$. The attainable set of z_0 in Ω is

$$\begin{aligned} \mathcal{A}_{z_0}(\Omega) &= \{z \in \Omega \mid \text{there exists an } \mathcal{L}\text{-admissible path } \gamma \text{ such that} \\ &\quad \gamma(0) = z_0, \gamma(T) = z \text{ and } \gamma(\tau) \in \Omega \text{ for } 0 \leq \tau \leq T\}. \end{aligned}$$

The following version of the Harnack inequality is based on the definition of attainable set. It has been introduced in [10, 9] and in its general form in [19] for operators in the form (1.6).

Theorem 3.2 *Let u be a non negative solution of $\mathcal{L}u = 0$ in some bounded open set $\Omega \subset \mathbb{R}^{N+1}$, and let $z_0 \in \Omega$. Suppose that $\text{Int}(\overline{\mathcal{A}_{z_0}(\Omega)}) \neq \emptyset$. Then, for every compact set $K \subset \text{Int}(\overline{\mathcal{A}_{z_0}(\Omega)})$ there exists a positive constant C_K , only depending on Ω, K, z_0 and \mathcal{L} , such that*

$$\sup_K u(z) \leq C_K u(z_0).$$

If the operator \mathcal{L} is also invariant with respect to suitable non Euclidean *translations* and *dilations*, then Theorem 3.2 restores an *invariant Harnack inequality* useful for the construction of Harnack chains.

HYPOTHESIS [G1] *There exists a Lie group $\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ such that X_1, \dots, X_m, Y are left invariant on \mathbb{G} , i.e.: given $\xi \in \mathbb{R}^{N+1}$ and denoting by $\ell_\xi(z) = \xi \circ z$, the left translation of $z \in \mathbb{R}^{N+1}$ it holds*

$$\begin{aligned} X_i(u(\ell_\xi(z))) &= (X_i u)(\ell_\xi(z)), \quad i = 1, \dots, m, \\ Y(u(\ell_\xi(z))) &= (Y u)(\ell_\xi(z)), \end{aligned}$$

for every smooth function u .

As we will see in the next Sections, all the examples listed in the Introduction do satisfy the above assumption, that replaces the usual invariance with respect to the Euclidean translation. For some operators \mathcal{L} considered in this note, the vector fields X_1, \dots, X_m, Y are also invariant

with respect to a rescaling property $(\delta_\lambda)_{\lambda>0}$ of the Lie group \mathbb{G} , which replaces the multiplication by a positive scalar in a vector space.

HYPOTHESIS [G2] *There exists a dilation $(\delta_\lambda)_{\lambda>0}$ on the Lie group \mathbb{G} such that the vector fields X_1, \dots, X_m are δ_λ -homogeneous of degree one and Y is δ_λ -homogeneous of degree two. i.e.:*

$$\begin{aligned} X_i(u(\delta_\lambda(z))) &= \lambda(X_i u)(\delta_\lambda(z)) \\ Y(u(\delta_\lambda(z))) &= \lambda^2(Yu)(\delta_\lambda(z)) \end{aligned}$$

for every smooth function u .

When both of assumptions [G1] and [G2] are satisfied, we say that $\mathbb{G} = (\mathbb{R}^{N+1}, \circ, (\delta_\lambda)_{\lambda>0})$ is a *homogeneous* Lie group, and the operator \mathcal{L} is invariant with respect to the left translations of \mathbb{G} , and homogeneous of degree 2 with respect to the dilation of \mathbb{G} . In this case we easily obtain from Theorem 3.2 an invariant Harnack inequality analogous to Corollary 2.3. Consider any bounded open set $\Omega \subset \mathbb{R}^{N+1}$ with $0 \in \Omega$ and suppose that it is *star-shaped* with respect to $(\delta_\lambda)_{\lambda>0}$, that is

$$\delta_r(\Omega) := \{\delta_r(z) \mid z \in \Omega\} \subset \Omega, \quad \text{for every } r \in]0, 1].$$

If $\text{Int}(\overline{\mathcal{A}_0(\Omega)}) \neq \emptyset$, we choose any compact set $K \subset \text{Int}(\overline{\mathcal{A}_0(\Omega)})$. For every $r > 0$ and $z_0 \in \mathbb{R}^{N+1}$ we set

$$\Omega_r(z_0) = z_0 \circ \delta_r(\Omega) := \{z_0 \circ \delta_r(z) \mid z \in \Omega\}.$$

Note that we also have $z_0 \circ \delta_\rho(K) \subset \text{Int}(\overline{\mathcal{A}_{z_0}(\Omega_r(z_0))})$ for every $\rho \in]0, r]$, since Ω is star-shaped. We define

$$\mathcal{P}_r(z_0) = \bigcup_{0 < \rho \leq r} z_0 \circ \delta_\rho(K).$$

Theorem 3.3 *Let \mathcal{L} be an operator in the form (1.6) satisfying assumptions [G1] and [G2] and let $\Omega_r(z_0)$ as above. Suppose that $\text{Int}(\overline{\mathcal{A}_{z_0}(\Omega_r(z_0))}) \neq \emptyset$, then*

$$\sup_{\mathcal{P}_r(z_0)} u(x, y, t) \leq C_K u(z_0)$$

for every positive solution u of $\mathcal{L}u = 0$ in $\Omega_r(z_0)$. Here C_K is the same constant appearing in Theorem 3.2.

Theorem 3.3 is the Harnack inequality that replaces Corollary 2.3 in the non Euclidean setting that is natural for the study of degenerate operators \mathcal{L} . In accordance with Remark 2.5, this is the first ingredient for the construction of Harnack chains. It turns out that the second ingredient is the \mathcal{L} -admissible path, which is the natural substitute of the segment used in the Euclidean setting. To replicate the construction made in the proof of Theorem 2.4 we only need to choose γ , with $\gamma(0) = (x_0, t_0)$, and $\mathcal{P}_{(x_0, t_0)}$ with the following property:

$$\text{there exists } s_0 \in]0, t_0 - t[\text{ such that } \gamma(s) \in \mathcal{P}_{(x_0, t_0)} \text{ for } s \in]0, s_0]. \quad (3.1)$$

All the examples in this note satisfy (3.1). Thus we have what we need to construct a Harnack chain $\{(x_0, t_0), (x_1, t_1), \dots, (x_k, t_k), (x, t)\}$ with starting point at (x_0, t_0) and end point at (x, t) .

In order to find an accurate bound of the positive solutions of $\mathcal{L}u = 0$ we need to control the *length* k of the Harnack chain. It is possible to prove that there exists a positive constant h such that, if we construct the Harnack chain by using the \mathcal{L} -admissible path γ as in Definition 3.1, with $z_0 = (x_0, t_0)$ and $z = (x, t)$, then $T = t_0 - t$ and we have

$$k \leq \frac{1}{h}\Phi(\omega) + 1, \quad \Phi(\omega) := \int_0^{t_0-t} \|\omega(s)\|^2 ds. \quad (3.2)$$

In the sequel we will refer to the integral appearing in (3.2) as the *cost* of the path γ associated to The *control* $(\omega_1, \dots, \omega_m)$. We then conclude that there exist three positive constants θ, h and M , with $\theta < 1$ and $M > 1$, only depending on the operator \mathcal{L} such that

$$u(x, t) \leq M^{1+\frac{\Phi(\omega)}{h}} u(x_0, t_0), \quad (3.3)$$

for every positive solution u of $\mathcal{L}u = 0$, where $(x_0, t_0), (x, t) \in \mathbb{R}^N \times]0, T[$ are such that $0 < t_0 - t < \theta t_0$.

Note that (3.3) provides us with a bound depending on the choice of the \mathcal{L} -admissible path γ steering (x_0, t_0) to (x, t) . In order to get the best exponent, we can optimize the choice of γ . With this spirit, we define the *Value function*

$$\Psi(x_0, t_0, x, t) = \inf \{ \Phi(\omega) \}, \quad (3.4)$$

where the *infimum* is taken in the set of all the \mathcal{L} -admissible paths γ steering (x_0, t_0) to (x, t) , and satisfying (3.1). We summarize this construction in the following general statement.

Let \mathcal{L} be an operator in the form (1.6) satisfying conditions [H], [G1] and [G2], and assume that there is a positive r and an open star-shaped set Ω with $0 \in \Omega$ such that $\text{Int}(\overline{\mathcal{A}_0(\Omega_r(0))}) \neq \emptyset$. If moreover all the \mathcal{L} -admissible paths γ steering (x_0, t_0) to (x, t) satisfy (3.1), then there exist three positive constants θ, h and M , with $\theta < 1$ and $M > 1$, only depending on the operator \mathcal{L} such that the following property holds.

Let $(x_0, t_0), (x, t) \in \mathbb{R}^{N+1}$ with $0 < t_0 - t < \theta t_0$. Then, for every positive solution $u : \mathbb{R}^N \times]0, T[$ of $\mathcal{L}u = 0$ it holds

$$u(x, t) \leq M^{1+\frac{1}{h}\Psi(x_0, t_0, x, t)} u(x_0, t_0). \quad (3.5)$$

The inequality (3.5) is the main step in the proof of our lower bound for the fundamental solution. All the examples considered in this note satisfy conditions [H], [G1]. Some examples also satisfy [G2], some examples do not. However, in this case, a scale invariant Harnack inequality still holds true, then the method provides us with a lower bound of the fundamental solution.

4 Degenerate hypoelliptic operators on homogeneous groups

The Heat operator on the Heisenberg group

$$\mathcal{L} = X_1^2 + X_2^2 - \partial_t$$

where

$$X_1 = \partial_x - \frac{1}{2}y\partial_w, \quad X_2 = \partial_y + \frac{1}{2}x\partial_w$$

are vector fields acting on the variable $(x, y, w, t) \in \mathbb{R}^4$, is the simplest example of degenerate operator built by a sub-Laplacian on a stratified Lie group. The vector fields X_1, X_2 are invariant with respect to the left translation on the Heisenberg group on \mathbb{R}^3 , whose operation is defined as

$$(x_0, y_0, w_0) \circ (x, y, w) = (x_0 + x, y_0 + y, w_0 + w + \frac{1}{2}(x_0y - y_0x)).$$

The above operation is extended to \mathbb{R}^4 by setting

$$(x_0, y_0, w_0, t_0) \circ (x, y, w, t) = (x_0 + x, y_0 + y, w_0 + w + \frac{1}{2}(x_0y - y_0x), t_0 + t).$$

Moreover \mathcal{L} is invariant with respect to the following dilation

$$\delta_r(x, y, w, t) = (rx, ry, r^2w, r^2t),$$

then the hypotheses [G1] and [G2] are fulfilled by \mathcal{L} . Furthermore, it satisfies the following property.

[C] *For every $x_0, x \in \mathbb{R}^N$, and for every positive T there exists an absolutely continuous path $\gamma_0 : [0, T] \rightarrow \mathbb{R}^N$ such that*

$$\dot{\gamma}_0(\tau) = \sum_{k=1}^m \omega_k(\tau) X_k(\gamma_0(\tau)), \quad \gamma_0(0) = x_0, \gamma_0(T) = x. \quad (4.1)$$

Note that, for operators \mathcal{L} in the form (1.6) with $X_0 = 0$, condition [C] is equivalent to the *strong Hörmander condition*

$$\text{rank Lie}\{X_1, \dots, X_m\}(x) = N, \quad \forall x \in \mathbb{R}^N,$$

Moreover, for every $\Omega \subset \mathbb{R}^{N+1}$ and for every $(x_0, t_0) \in \Omega$ there exist a positive ε and a neighborhood U of x_0 such that $U \times]t_0, t_0 + \varepsilon[\subset \mathcal{A}_{(x_0, t_0)}(\Omega)$. This particular geometric property of the attainable set implies that an invariant Harnack inequality analogous to the standard parabolic one holds for this operator. The only difference is that the Euclidean translation and the parabolic dilations are replaced by the operations used to satisfy hypotheses [G1] and [G2]. In conclusion, the hypotheses we need to prove (3.5) are satisfied by the heat operator on the Heisenberg group. In particular, this method leads us to the lower bound of the following version of (1.5): there exist positive constants c^-, C^-, c^+, C^+ such that

$$\frac{c^-}{(t - \tau)^{N/2}} \exp\left(-C^- \frac{d_{CC}(x, \xi)^2}{t - \tau}\right) \leq \Gamma(x, t, \xi, \tau) \leq \frac{C^+}{(t - \tau)^{N/2}} \exp\left(-c^+ \frac{d_{CC}(x, \xi)^2}{t - \tau}\right), \quad (4.2)$$

where d_{CC} denotes the *Carnot-Caratheodory distance*

$$d_{CC}(x_0, x) = \inf\{\ell(\gamma_0) \mid \gamma_0 \text{ is as in (4.1)}\}, \quad \ell(\gamma) := \int_0^T \|\omega(s)\| ds.$$

We recall that the upper bound was proved by Davies in [12], and the upper and lower bounds are due to Jerison and Sánchez-Calle [17], and to Varopoulos, Saloff-Coste and Coulhon [38].

Note that $\Psi(x_0, t_0, x, t) = \frac{d_{CC}(x_0, x)^2}{t_0 - t}$. Indeed, if we consider the path $\gamma(s) = (\gamma_0(s), t_0 - s)$ with $0 \leq s \leq t_0 - t$, then by the Hölder inequality, we obtain $\ell(\gamma_0) \leq \sqrt{\Phi(\omega)}\sqrt{t_0 - t}$. Moreover the equality occurs only if the norm of the control ω is constant, that is

$$\ell(\gamma_0) = \sqrt{\Phi(\omega)}\sqrt{t_0 - t} \iff (\omega_1^2 + \dots + \omega_m^2)(s) = \frac{\Phi(\omega)}{t_0 - t} \text{ for every } s \in [0, t_0 - t].$$

We refer to the article [7] for the study of a more general class of operator satisfying [G1], [G2] and [C], that includes heat operators on Carnot groups and also operators \mathcal{L} with $X_0 \neq 0$. We also recall that in the article [11] the analogous upper bound has been proved by using a PDE method combined with the Optimal Control Theory.

5 Degenerate Kolmogorov equations

The simplest degenerate example of degenerate Kolmogorov operator is

$$\mathcal{L} := \partial_x^2 + x\partial_y - \partial_t, \quad (x, y, t) \in \mathbb{R}^2 \times]0, T[, \quad (5.1)$$

it writes in the form (1.6), if the vector fields X, Y are

$$X(x, y, t) = \partial_x \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y(x, y, t) = x\partial_y - \partial_t \sim \begin{pmatrix} 0 \\ x \\ -1 \end{pmatrix}.$$

\mathcal{L} is related to the following stochastic process

$$\begin{cases} X_t = x_0 + W_t, \\ Y_t = y_0 + \int_0^t (x_0 + W_s) ds. \end{cases}$$

which satisfies the *Langevin equation* $dX_t = dW_t, dY_t = X_t dt$. We recall that this kind of stochastic process appears in several research areas. For instance, in Kinetic Theory, $(X_t)_{t \geq 0}$ describes the velocity of a particle, while $(Y_t)_{t \geq 0}$ is its position. We note that

i) \mathcal{L} is invariant with respect to the operation

$$(x_0, y_0, t_0) \circ (x, y, t) = (x + x_0, y + y_0 - tx_0, t + t_0), \quad (x, y, t), (x_0, y_0, t_0) \in \mathbb{R}^3,$$

ii) \mathcal{L} is homogeneous of degree 2 with respect to the dilation

$$(\delta_\rho)_{\rho > 0} : (x, y, t) \mapsto (\rho x, \rho^3 y, \rho^2 t),$$

iii) The \mathcal{L} -admissible paths are the solutions $\gamma(s) = (x(s), y(s), t(s))$ of the following equation

$$\begin{cases} \dot{x}(s) = \omega(s), & x(0) = x_0, \\ \dot{y}(s) = x(s), & y(0) = y_0, \\ \dot{t}(s) = -1, & t(0) = t_0. \end{cases}$$

It is easy to check that the attainable set of the point $(0, 0, 0)$ in the open set $\Omega =]-1, 1[^3$ is $\mathcal{A}_{(0,0,0)}(\Omega) = \{(x, y, t) \in \Omega \mid t < -|y|\}$, (see Fig. 4).

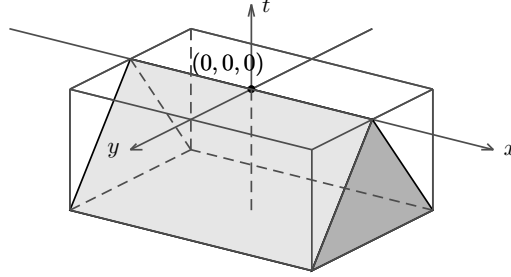


FIG. 4 - $\mathcal{A}_{(0,0,0)}(\Omega)$.

As the interior of $\mathcal{A}_{(0,0,0)}(\Omega)$ is not empty, Theorem 3.3 gives an invariant Harnack inequality for \mathcal{L} , and we can apply (3.5) to prove lower bounds for positive solutions defined on the domain $\mathbb{R}^2 \times]0, T[$. The Optimal Control Theory provides us with an explicit expression of the value function Ψ for \mathcal{L}

$$\Psi(x_0, y_0, t_0; x, y, t) = \frac{(x - x_0)^2}{t - t_0} + \frac{12}{(t - t_0)^3} \left(y - y_0 - (t - t_0) \frac{(x + x_0)}{2} \right)^2.$$

This is a remarkable fact, as it is known that the explicit expression of the fundamental solution of \mathcal{L} was written by Kolmogorov (1934) and is

$$\Gamma(x, y, t, x_0, y_0, t_0) = \frac{\sqrt{3}}{2\pi(t - t_0)^2} \exp \left(-\frac{(x - x_0)^2}{4(t - t_0)} - \frac{3}{(t - t_0)^3} \left(y - y_0 - (t - t_0) \frac{(x + x_0)}{2} \right)^2 \right).$$

The lower bound based on the value function Ψ is useful as we consider Kolmogorov equations in the form

$$\partial_t u(x, t) = \sum_{i,j=1}^m a_{ij}(x, t) \partial_{x_i x_j}^2 u(x, t) + \sum_{i,j=1}^N b_{ij} x_j \partial_{x_i} u(x, t), \quad (x, t) \in \mathbb{R}^N \times]0, T], \quad (5.2)$$

with bounded Hölder continuous coefficients a_{ij} 's. In the study of this family of operators, we assume that $m < N$, the matrix $(a_{ij}(t, x))_{i,j=1,\dots,m}$ is uniformly positive in \mathbb{R}^m . Moreover, the Hörmander condition is satisfied for the operator $\mathcal{L}_{(x_0, t_0)}$ frozen at some point $(x_0, t_0) \in \mathbb{R}^{N+1}$, that is obtained from the equation in (5.2) by replacing every function $a_{ij} = a_{ij}(x, t)$ with $a_{ij}(x_0, t_0)$. It turns out that this condition does not depend on the choice of the point (x_0, t_0) , that $\mathcal{L}_{(x_0, t_0)}$ is invariant with respect to a Lie group \mathbb{G} on \mathbb{R}^{N+1} which does not depend on (x_0, t_0) . In this case the *parametrix method* provides us with the existence of a fundamental solution Γ of the operator introduced in (5.2). The method also gives an upper bound of the form

$$\Gamma(x, t; x_0, t_0) \leq \frac{C^+}{(t - t_0)^{Q/2}} \exp(-c^+ \Psi(x, t; x_0, t_0)) \quad (x_0, t_0), (x, t) \in \mathbb{R}^N \times]0, T],$$

where Q is the *homogeneous dimension* of the space \mathbb{R}^N with respect to the underlying Lie Group in \mathbb{R}^{N+1} , and C^+, c^+ are constants depending on the operator and on T . The method described in this Section gives the analogous lower bound for Γ

$$\frac{c^-}{(t-t_0)^{Q/2}} \exp(-C^- \Psi(x, t; x_0, t_0)) \leq \Gamma(x, t; x_0, t_0) \quad (x_0, t_0), (x, t) \in \mathbb{R}^N \times]0, T].$$

We recall that the parametrix method has been used by several authors for the study of degenerate Kolmogorov equations. We recall the works of Weber [39], Il'In [16], Sonin [36], Polidoro [32, 33], Di Francesco and Polidoro [14]. In particular, the lower bound of the fundamental is proved in [33] and in [14].

More recently, Delarue and Menozzi [13] weaken the regularity assumptions on the coefficients a_{ij} 's and obtain analogous bounds by combining stochastic control methods with the parametrix representation of the fundamental solution given by McKean and Singer in [24].

6 More degenerate equations

In this Section we consider a stochastic process studied By Cinti, Menozzi and Polidoro in [9]. It is similar to the one considered in Section 4, as it writes as follows

$$\mathcal{L} := \partial_x^2 + x^2 \partial_y - \partial_t, \quad (x, y, t) \in \mathbb{R}^2 \times (0, T), \quad (6.1)$$

and is related to the following stochastic differential equation

$$\begin{cases} X_t = x_0 + W_t, \\ Y_t = y_0 + \int_0^t (x_0 + W_s)^2 ds. \end{cases} \quad (6.2)$$

A representation of the density of this process has been obtained from the seminal works of Kac [18] in terms of the Laplace transform of the process $(Y_t)_{t \geq 0}$. We also refer to the monograph of Borodin and Salminen [6] for an expression in terms of special functions. We also quote the works of Smirnov [35] and Tolmatz [37] on the distribution function of the square of the Brownian bridge.

We give explicit upper and lower bounds for the density of the process $(X_t, Y_t)_{t \geq 0}$ by the approach described in Section 3. Note that new difficulties appear in the study of the operator \mathcal{L} defined in (6.1). Indeed, if we write \mathcal{L} as follows

$$\mathcal{L} = X^2 + Y, \quad \text{with} \quad X = \partial_x, \quad Y = x^2 \partial_y - \partial_t,$$

then the commutator $[X, Y](x, y, t) = 2x \partial_y$ vanishes in the set $\{x = 0\}$, and we need a second commutator $[X, [X, Y]](x, y, t) = 2 \partial_y$ to satisfy the Hörmander condition at every point of \mathbb{R}^3 . As a consequence, a Lie group leaving invariant the equation $\mathcal{L}u = 0$ cannot exist. This problem is overcome by a *lifting procedure*. Specifically, we consider the following operator

$$\widetilde{\mathcal{L}} := \partial_x^2 + x \partial_w + x^2 \partial_y - \partial_t, \quad (x, y, w, t) \in \mathbb{R}^3 \times (0, T),$$

and we consider any solution of $\mathcal{L}u = 0$ as a function that does not depend on w , and that solves the equation $\widetilde{\mathcal{L}}u = 0$. The lifting procedure allows us to rely on the Lie group invariance of $\widetilde{\mathcal{L}}$ in the study of the positive solutions of $\mathcal{L}u = 0$. Indeed, we have

i) The operator $\widetilde{\mathcal{L}}$ is invariant with respect to the following Lie group operation

$$(x_0, y_0, w_0, t_0) \circ (x, y, w, t) = (x + x_0, y + y_0 + 2x_0w - tx_0^2, w + w_0 - tx_0, t + t_0),$$

defined for every $(x, y, w, t), (x_0, y_0, w_0, t_0) \in \mathbb{R}^4$. In particular, it holds

$$(\widetilde{\mathcal{L}}u)(z_0 \circ z) = \widetilde{\mathcal{L}}(u(z_0 \circ z)),$$

for every $z_0 = (x_0, y_0, w_0, t_0)$ and $z = (x, y, w, t) \in \mathbb{R}^4$.

ii) The operator $\widetilde{\mathcal{L}}$ is invariant with respect to the following dilation

$$(\delta_\rho)_{\rho \geq 0} : (x, y, w, t) \mapsto (\rho x, \rho^4 y, \rho^3 w, \rho^2 t)$$

that is it holds

$$\rho^2 (\mathcal{L}u)(\rho x, \rho^3 y, \rho^2 t) = \mathcal{L}(u(\rho x, \rho^3 y, \rho^2 t)).$$

iii) The attainable set of the *origin* in the box $\Omega =]-1, 1[^4$ is

$$\mathcal{A}_{(0,0,0,0)}(\Omega) = \left\{ (x, w, y, t) \in]-1, 1[^4 \mid 0 \leq y \leq -t, w^2 \leq -ty \right\}.$$

Figure 4 describes the projection on the hyperplane $\{x = 0\}$ of the set $\mathcal{A}_{(0,0,0,0)}$

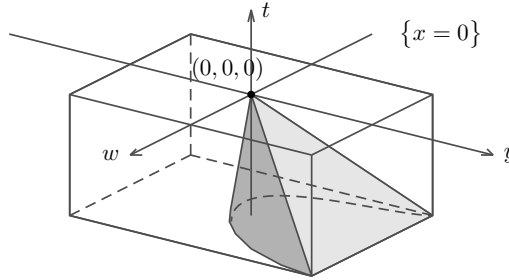


FIG. 5 - PROJECTION OF $\mathcal{A}_{(0,0,0,0)}(\Omega)$ ON THE SET $\{x = 0\}$.

Then an invariant Harnack inequality needed to construct Harnack chains for the positive solutions of $\widetilde{\mathcal{L}}u = 0$ is available. The main result of the article [9] is the following

Theorem 6.1 *Let Γ denote the fundamental solution of $\partial_{xx} + x^2 \partial_y - \partial_t$.*

- *If $\eta - y \leq 0$, then $\Gamma(x, y, t, \xi, \eta, \tau) = 0$;*
- *if $\frac{\eta - y}{(t - \tau)^2} > \frac{x^2 + \xi^2}{t - \tau} + 1$, then*

$$\Gamma(x, y, t, \xi, \eta, \tau) \approx \frac{1}{(t - \tau)^{5/2}} \exp \left(-C \left(\frac{(x - \xi)^2}{t - \tau} + \frac{\eta - y}{(t - \tau)^2} \right) \right);$$

- if $0 < \frac{\eta-y}{(t-\tau)^2} < \frac{1}{2}$, then

$$\Gamma(x, y, t, \xi, \eta, \tau) \approx \frac{1}{(t-\tau)^{5/2}} \exp \left(-C \left(\frac{x^4 + \xi^4 + (t-\tau)^2}{\eta-y} \right) \right).$$

We conclude this section with some remarks. We first note that, because of the particular form of the attainable set $\mathcal{A}_{(0,0,0,0)}(\Omega)$, it is not true that all the \mathcal{L} -admissible paths γ steering z_0 to z satisfy (3.1). For this reason, in the proof of our main result we do not solve any optimal control problem. We prove our lower bound by choosing smart admissible paths. This construction does not guarantee the optimality of the lower bounds. However, the comparison with the upper bound, that has the same asymptotic behavior, shows the optimality of both of them. The diagonal bounds and the upper bounds have been obtained by using probabilistic methods, and Malliavin Calculus in particular.

We eventually recall that more general operators and stochastic processes are studied in [9]. Precisely, we consider for every positive integer k the process $(X_t, Y_t)_{t \geq 0}$, with value in $\mathbb{R}^n \times \mathbb{R}$

$$\begin{cases} X_t = x + W_t \\ Y_t = y + \int_0^t \sum_j (x + W_s)_j^k ds \end{cases}$$

whose Kolmogorov equation is

$$\mathcal{L} := \frac{1}{2} \Delta_x + (x_1^k + \dots + x_n^k) \partial_y - \partial_t$$

and

$$\begin{cases} X_t = x + W_t, & (k \text{ even}) \\ Y_t = y + \int_0^t |x + W_s|^k ds \end{cases}$$

whose Kolmogorov equation is

$$\mathcal{L} := \frac{1}{2} \Delta_x + |x|^k \partial_y - \partial_t$$

We refer to the article [9] for the precise statement of our achievements and for further details.

7 Operators related to Arithmetic Average Asian Options

In this section we consider the operator

$$\mathcal{L} = x^2 \partial_{xx} + x \partial_x + x \partial_y - \partial_t$$

with $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times (0, T)$. It appears in the Black and Scholes setting when we consider the pricing problem for Arithmetic Average Asian Option. Specifically, we assume that the price of an asset $(X_t)_{t \geq 0}$ is described by a geometric Brownian Motion, and that the option depends on the arithmetic average of $(X_t)_{t \geq 0}$. Then, according to the Black and Scholes theory, the value of the option v is modeled by a function $v = v(t, X_t, Y_t)$ where

$$\begin{cases} X_t = x_0 e^{\sqrt{2} W_t}, \\ Y_t = y_0 + x_0 \int_0^t e^{\sqrt{2} W_s} ds. \end{cases}$$

This system was widely studied by Yor who wrote in [40] its joint density

$$p(x, y, t) = \frac{1}{2\sqrt{xy^2}} \frac{e^{\frac{\pi^2}{t}}}{\pi\sqrt{\pi t}} \exp\left(-\frac{1+x}{2y}\right) \psi\left(\frac{\sqrt{x}}{y}, \frac{t}{2}\right), \quad (7.1)$$

where

$$\psi(z, t) = \int_0^\infty e^{-\frac{\xi^2}{2t}} e^{-z \cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi. \quad (7.2)$$

As in the previous example, the density of the stochastic process $(X_t, Y_t)_{t \geq 0}$ is not strictly positive in the whole set $\mathbb{R}^+ \times \mathbb{R} \times (0, T)$. In particular, its support is $\mathbb{R}^+ \times (y_0, +\infty) \times (t_0, T)$.

Monti and Pascucci observe in [26] that \mathcal{L} is invariant with respect to the following group operation on $\mathbb{R}^+ \times \mathbb{R}^2$:

$$(x_0, y_0, t_0) \circ (x, y, t) = (x_0 x, y_0 + x_0 y, t_0 + t). \quad (7.3)$$

Indeed, if we set

$$v(x, y, t) = u(x_0 x, y_0 + x_0 y, t_0 + t), \quad (7.4)$$

then $\mathcal{L}v = 0$ if, and only if $\mathcal{L}u = 0$.

Note that \mathcal{L} is not invariant with respect to any dilation group $(\delta_\rho)_{\rho \geq 0}$. On the other hand, as

$$\mathcal{L} = X^2 + Y, \quad \text{with} \quad X(x, y, t) = x\partial_x, \quad Y(x, y, t) = x\partial_y - \partial_t,$$

we have that \mathcal{L} can be approximated by the Kolmogorov operator (5.1) defined in section 5. Indeed, we can consider the coefficient x of the vector field X as a smooth function that is bounded and bounded by below on every compact set $K \subset \mathbb{R}^+ \times \mathbb{R} \times (0, T)$. For this reason, the Harnack inequality introduced in Section 5 also applies to \mathcal{L} .

The \mathcal{L} admissible paths are the solutions of the following differential equation

$$\begin{cases} \dot{x}(s) = \omega(s)x(s), & x(0) = x_0, \\ \dot{y}(s) = x(s), & y(0) = y_0, \\ \dot{t}(s) = -1, & t(0) = t_0, \end{cases}$$

and we denote by $\Psi(x_0, y_0, t_0, x, y, t)$ the value function of the relevant optimal control problem with quadratic cost. The main result for the fundamental solution $\Gamma(x, y, t; x_0, y_0, t_0)$ of the operator \mathcal{L} is the following

Theorem 7.1 *Let Γ be the fundamental solution of \mathcal{L} . Then for every $(x_0, y_0, t_0), (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$ we have*

$$\Gamma(x, y, t, x_0, y_0, t_0) = 0 \quad \forall (x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2 \setminus \{]-\infty, y_0[\times]t_0, T[\}. \quad (7.5)$$

Moreover, for arbitrary $\varepsilon \in]0, 1[$, there exist two positive constants $c_\varepsilon^-, C_\varepsilon^+$ depending on ε , on T and on the operator \mathcal{L} , and two positive constants C^-, c^+ , only depending on the operator \mathcal{L} such that

$$\begin{aligned} \frac{c_\varepsilon^-}{x_0^2(t-t_0)^2} \exp\left(-C^- \Psi(x, y + x_0\varepsilon(t-t_0), t - \varepsilon(t-t_0); x_0, y_0, t_0)\right) \leq \\ \Gamma(x, y, t; x_0, y_0, t_0) \leq \\ \frac{C_\varepsilon^+}{x_0^2(t-t_0)^2} \exp\left(-c^+ \Psi(x, y - x_0\varepsilon, t + \varepsilon; x_0, y_0, t_0)\right), \end{aligned} \quad (7.6)$$

for every $(x, y, t) \in \mathbb{R}^+ \times]-\infty, y_0 - x_0 \varepsilon(t - t_0)[\times]t_0, T]$.

Clearly, the knowledge of the function Ψ is crucial for the application of our Theorem 7.1. In [8], it is shown that one can write the function Ψ in terms of the function g defined as follows

$$g(r) = \begin{cases} \frac{\sinh(\sqrt{r})}{\sqrt{r}}, & r > 0, \\ 1, & r = 0, \\ \frac{\sin(\sqrt{-r})}{\sqrt{-r}}, & -\pi^2 < r < 0, \end{cases}$$

and it is proven the following proposition

Proposition 7.2 *For every $(x, y, t), (x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$, with $t_0 < t$ and $y_0 > y$, we have*

$$\begin{cases} \Psi(x_1, y_1, t_1; x_0, y_0, t_0) = E(t_1 - t_0) + \frac{4(x_1 + x_0)}{y_0 - y_1} - 4\sqrt{E + \frac{4x_1x_0}{(y_0 - y_1)^2}}, & \text{if } E \geq -\frac{\pi^2}{t_1 - t_0}; \\ \Psi(x_1, y_1, t_1; x_0, y_0, t_0) = E(t_1 - t_0) + \frac{4(x_1 + x_0)}{y_0 - y_1} + 4\sqrt{E + \frac{4x_1x_0}{(y_0 - y_1)^2}}, & \text{if } -\frac{4\pi^2}{t_1 - t_0} < E < -\frac{\pi^2}{t_1 - t_0}. \end{cases}$$

where

$$E = \frac{4}{(t - t_0)^2} g^{-1} \left(\frac{y_0 - y}{(t - t_0)\sqrt{x_0x}} \right).$$

Moreover,

$$\begin{aligned} \frac{\Psi(x, y, t; x_0, y_0, t_0)}{\frac{4}{(t - t_0)} \log^2 \left(\frac{y_0 - y}{(t - t_0)\sqrt{x_0x}} \right) + \frac{4(x_0 + x)}{y_0 - y}} &\rightarrow 1, & \text{as } \frac{y_0 - y}{(t - t_0)\sqrt{x_0x}} &\rightarrow +\infty; \\ \frac{\Psi(x, y, t; x_0, y_0, t_0)}{\frac{4(\sqrt{x} + \sqrt{x_0})^2}{y_0 - y} - \frac{4\pi^2}{(t - t_0)}} &\rightarrow 1, & \text{as } \frac{y_0 - y}{(t - t_0)\sqrt{x_0x}} &\rightarrow 0. \end{aligned}$$

As a final remark, we note that the above expression for the value function Ψ has been obtained by using the Pontryagin Maximum Principle [34], the upper bound is a consequence of the fact that Ψ satisfies the Hamilton-Jacobi-Bellman equation $Y\Psi + \frac{1}{4}(X\Psi)^2 = 0$.

The method described above also applies to the divergence form operator $\widetilde{\mathcal{L}}$ defined as

$$\widetilde{\mathcal{L}}u = x\partial_x(ax\partial_xu) + bx\partial_xu + x\partial_yu - \partial_tu,$$

where a and b are smooth bounded coefficients, with a bounded by below. Note that, in this case, an expression of Γ analogous to (7.1) is not available. We refer to the article [8] for the precise statement of the results of this section and for further details.

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